CIS677
Comments on homework 3

1. *In Lecture 6, some relations were claimed about the edge sets of some geometric graphs:*

$$RNG(S) \subset GG(S) \subset DG(S).$$

Prove these relations. Given efficient algorithms to compute $RNG(S)$ and $GG(S)$. (See problems 9.8 and 9.9 of the text for a hint.)

Let $a, b \in S$, and let c be the midpoint of segment $\overline{ab}$. Let $x = d(a, b)$, and let $B_a(x)$ denote the disk with center a and radius x; $B_b(x)$ denote the disk with center b and radius x, and $B_c(x/2)$ denote the disk with center c and radius $x/2$. The edge $e \equiv \{a, b\}$ is in $RNG(S)$ iff $B_a(x) \cap B_b(x)$ is empty (of sites), and in $GG(S)$ iff $B_c(x/2)$ is empty. Now e is an edge of DG(S) iff $VR(a, b)$ is nonempty, which is true iff there is some empty disk B with a and b on its bounding circle. But $B_c(x/2)$ has a and b on its bounding circle, so $e \in GG(S)$ implies $e \in DG(S)$. Now suppose $B_a(x) \cap B_b(x)$ is empty. Then if p is a point in $B_c(x/2)$, then $d(p, c) \leq x/2$, and since $d(c, a) = x/2$, by the triangle inequality

$$d(a, p) \leq d(p, c) + d(c, a) \leq x,$$

so $p \in B_a(x)$. Similarly $p \in B_b(x)$, so $B_c(x/2) \subset B_a(x) \cap B_b(x)$. Thus $e \in RNG(S)$ implies $B_a(x) \cap B_b(x)$ is empty, which implies $B_c(x/2)$ is empty, which implies $e \in GG(S)$.

Unfortunately, the text is wrong in its description of $RNG(S)$, and $RNG(S)$ is not at all trivial to compute in $O(n \log n)$, but fortunately no one seems to have gone crazy over this. My apologies for once again giving a misleading homework problem.

To compute $GG(S)$, compute $DG(S)$ and check each edge of $DG(S)$ to see if the appropriate disk is empty. To do this, check (in the notation above for a given edge $e = \{a, b\}$) if the edge e' of $VR(S)$ dual to e has $c \in e'$. This implies $B_c(x/2)$ is empty.

2. *Problem 7.10 of the text: Let P be a set of n points in the plane. Give an $O(n \log n)$ algorithm to find two points in P that are closest together. Show that your algorithm is correct.*

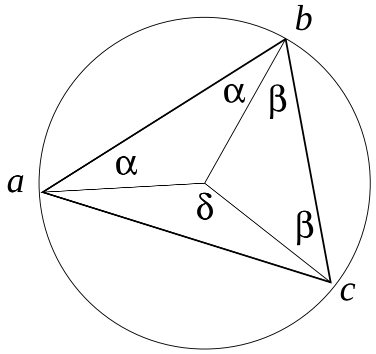
Suppose a, b are a closest pair of sites of P. Then with the notation for the previous problem, $B_a(x) \cup B_b(x)$ is empty (otherwise, some site is closer to a than b, or closer to b than a.) Since $B_a(x) \cap B_b(x) \subset B_a(x) \cup B_b(x)$, this implies $\{a, b\}$ is in $RNG(P)$, and so in $DG(P)$. The algorithm is therefore: compute $DG(P)$, and find a shortest edge of $DG(P)$. The endpoints of that edge are a closest pair of sites.

3. *Problem 7.11 of the text: Let P be be a set of n points in the plane. Given an $O(n \log n)$ time algorithm to find for each point p in P another point in P that is closest to it.*

   By an argument similar to that for the previous problem, for any site p, if q is the site closest to p (other than p itself), then $\{p, q\}$ is a Delaunay edge. Therefore, when searching for the site of P closest to p, it is sufficient to examine only the "Delaunay neighbors" of p, that is, those sharing a Delaunay edge with p. After computing $DG(P)$, compute the shortest edge incident to each $p \in P$; these edges yield the closest sites. Each edge is examined twice in this calculation, so the work is $O(n)$ after computing $DG(P)$.

4. *Problem 9.3 of the text.*

   Without loss of generality, assume that the circle is the unit circle centered at the origin. In the figure, a, b, c are unit vectors. We claim that the measure of $\angle abc$ is monotonically increasing in $d(a, c)$, and independent of the position of b on the circle. As a partial proof, note that the sum of the three angles at the center of the circle is 2π, and so



$$\delta + (\pi - 2\alpha) + (\pi - 2\beta) = 2\pi,$$

   using the fact that the sum of the angles of a triangle is π. Hence $\alpha + \beta = \delta/2$, and $\alpha + \beta$ is the measure of $\angle abc$. Moreover, $d(a, c) = 2 \sin \delta$, and so $d(a, c) = 2 \sin(2(\alpha + \beta))$, or

$$\alpha + \beta = \sin^{-1}(d(a, c)/2)/2,$$

   so the smaller $d(a, c)$ is, the smaller $\angle abc$ is: the length of the chord $d(a, b)$ is monotonically increasing in the (smaller) angle δ subtended by the chord.

   It follows that the smallest angle of a triangulation of a cocircular polygon is the angle opposite the shortest edge. However, the shortest edge is a bounding edge of the polygon: If a and c have some c' between them on the minor arc they determine, then $d(a, c) > d(a, c')$ and $d(a, c) > d(c, c')$, since the angles subtended by these chords are smaller. Hence the shortest edge is determined by consecutive points on the circle, and such consecutive points give edges of the polygon.

Since the smallest angle is opposite the shortest edge, and the shortest edge is a bounding edge of the polygon, and the edges of the polygon appear in every triangulation, the smallest angle is the same for every triangulation of the polygon.

5. *Use the farthest-point Voronoi diagram to show that the smallest circle enclosing a set of points can be found in $O(n \log n)$ time.*

   The center of the smallest enclosing circle is the point in the plane whose maximum distance to a site is minimum, over all points in the plane.

   If a point p is in the interior of a Voronoi region $VR(a)$ of the farthest-point Voronoi diagram, then moving from p towards a reduces the maximum distance to a site, until a Voronoi edge is reached. Therefore, for any point in the plane, there is a point on a Voronoi edge with maximum distance that is no larger. In other words, the center of the smallest enclosing circle is on some Voronoi edge. On a given Voronoi edge, equidistant to sites a and b, it is possible in $O(1)$ time to find the point on the edge with minimum distance to a and b. Such a point has smaller maximum distance than any other point on the edge. Therefore, it is sufficient to compute that minimum point for each edge, and find the minimum such point over all edges. This takes $O(1)$ time per edge, or $O(n)$ time overall, after the Voronoi diagram is computed in $O(n \log n)$ time.

   Note that the center need not be a vertex of the FPVD: the smallest enclosing circle may be determined by two points, and the center is on the Voronoi edge determined by them.

6. *The* discrete 1-center *of a set of points S is the member of S whose maximum distance to the remaining points of S is smallest, over all points of points of S. That is,* $\mathrm{cent}(S)$ *gives*

   $$\min_{p \in S} \max_{q \in S \setminus \{p\}} d(p, q).$$

   Show that the discrete 1-center of S can be found in $O(n \log n)$ time.

   Compute the farthest-point Voronoi diagram of S, and then build an optimal planar point location data structure for that diagram. For each $p \in S$, perform a point location operation to find the Voronoi region $VR(p')$ containing p; this gives the maximum distance $d(p, p')$ from p to another site. Find the maximum of these distances over all sites.