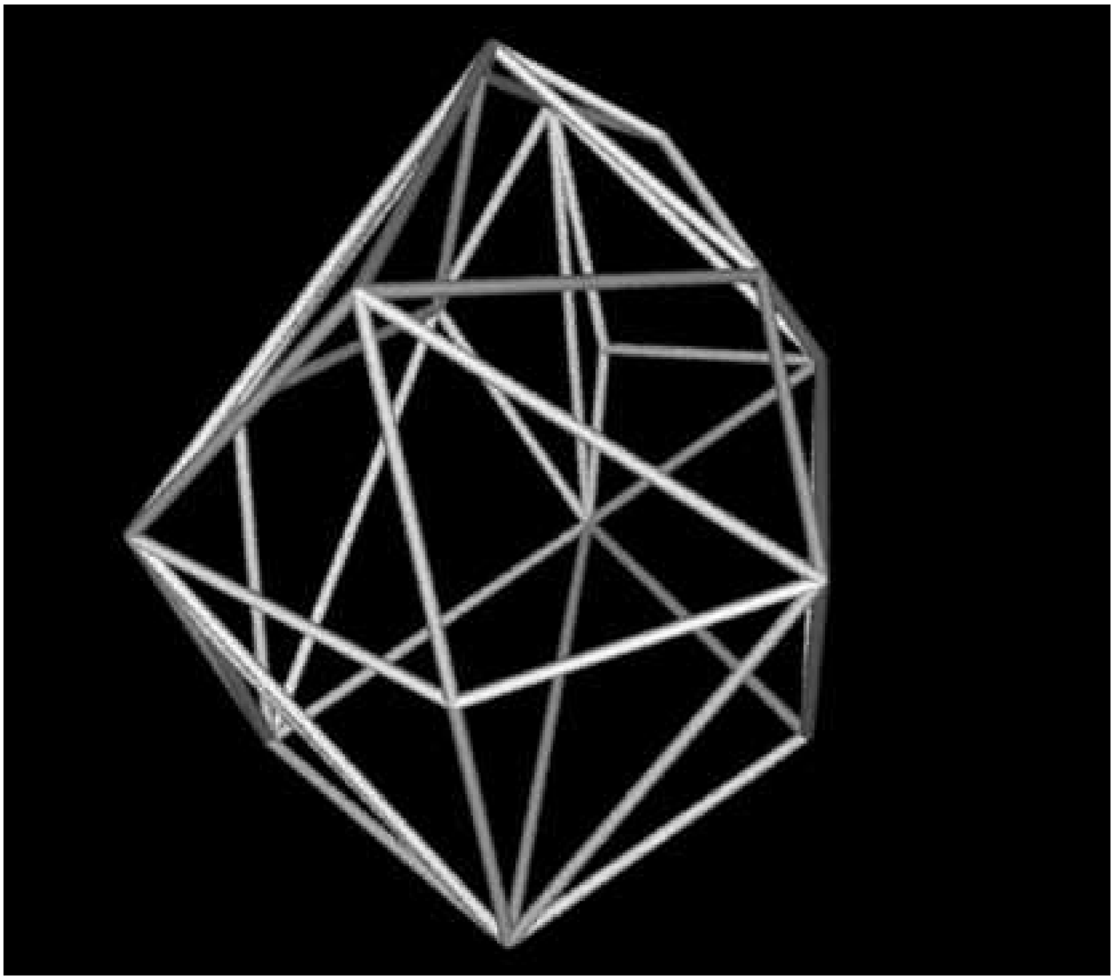

Convex Hulls: Some Algorithms and Applications

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Convex Hulls

- How to compute them:
  - in general dimension,
  - with arbitrary degeneracy,
  - exactly. (For reliability!)
- Why to compute them.

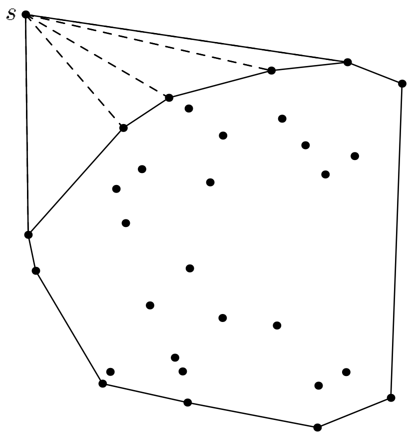
Related Constructions

- convex hulls;
- Voronoi diagrams;
- Delaunay triangulations;
- α-shapes;
- volumes of Voronoi regions;
- natural neighbor interpolants.

"discovery" :
Gram-Schmidt gives a simple numerical method for all.

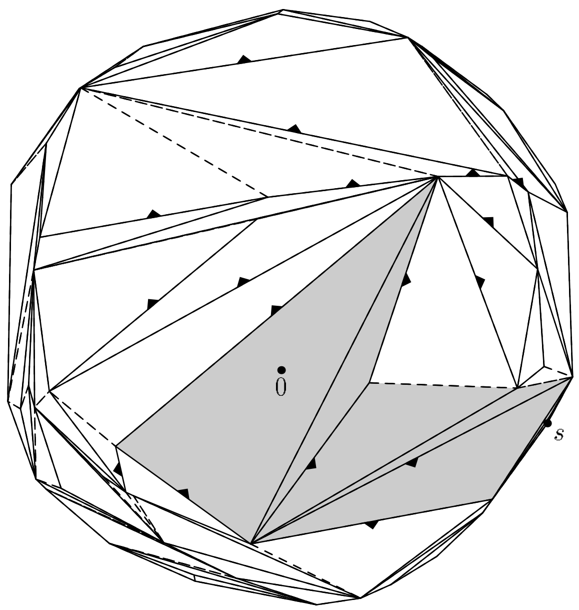
The (Combinatorial) Algorithm

- add sites one at a time,
- maintaining triangulation of hull.
- add new triangle for each facet *seen* by s;
- a new facet for each seen/unseen pair;



The Triangulation

- each simplex is array of $\{$vertex; opposite simplex$\}$;
- gives current state of hull;
- break ties so all facets are simplices;
- allows fast searching to find a visible facet;



The (Numerical) Algorithm

Problems:

1. Checking if a new site is in the flat of prior sites;

2. Finding a vector perpendicular to a facet.

Assume, by translation,
that flats pass through the origin.

Solve these by solving:

given vectors A=$\{a_1 \ldots a_{m-1}\}$,
and vector a_m, find component of $a_m \perp$ to A.

For (1), check if result is 0.

Gram-Schmidt

for $k := 1$ **upto** m
do
    $c_k := a_k$;
    for $j := k - 1$ **downto** 1
        $c_k -= c_j(c_k / c_j)$;
od

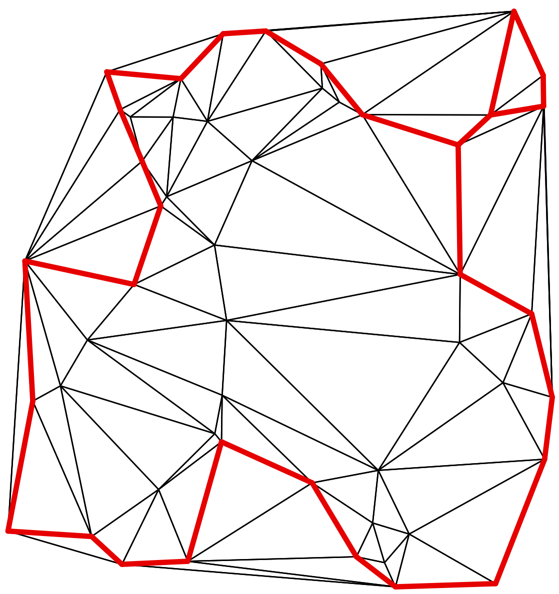
For exact arithmetic,
do additional "conditioning" steps as needed,
replacing a_k by $a_k - Za_j$
in exact arithmetic
when a_k angle to a_j small.

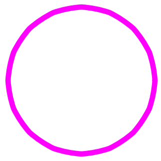
for $k := 1$ **upto** m
do
  loop forever
    $b_k := a_k$;
    for $j := k-1$ **downto** 1
      $b_k -= a_j(b_k/b_j)$;
    if $a_k^2 \leq 2b_k^2$
      then exit_loop;
    choose scaling factor s;
    $a_k *= s$;
    for $j := k-1$ **downto** 1
      $a_k -= a_j\lceil(a_k/b_j)\rfloor$;
  end_loop;
od

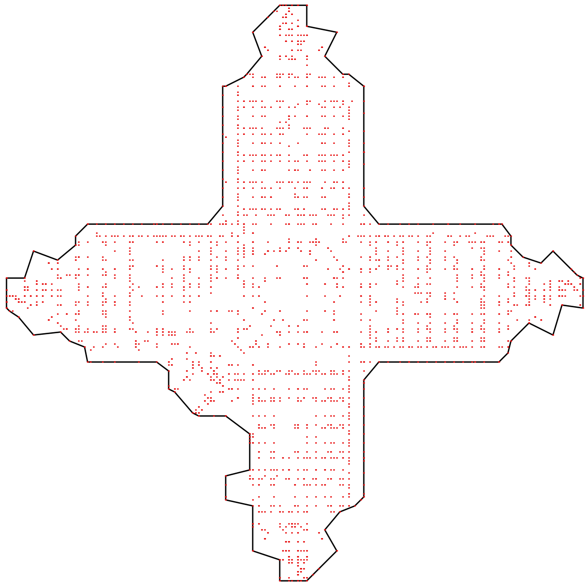
Alpha shapes

A subset of the faces of the D. triangulation:
Given r,
d-simplices with empty spheres of radius r
inscribed on their endpoints.

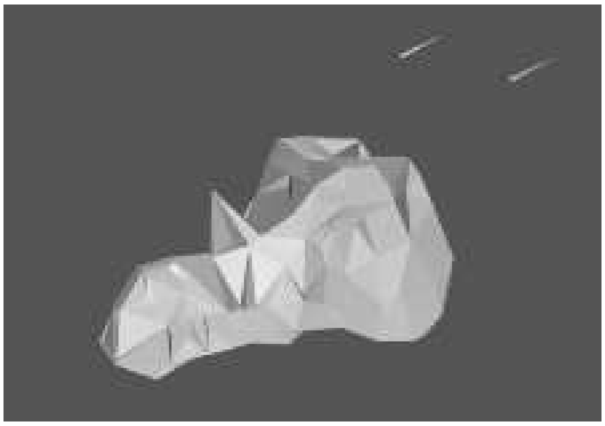
For $d = 2$:



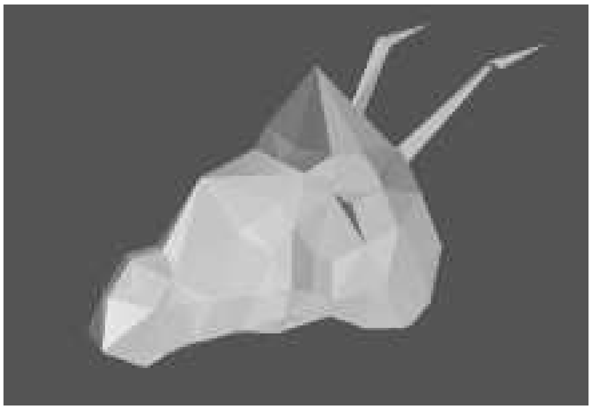




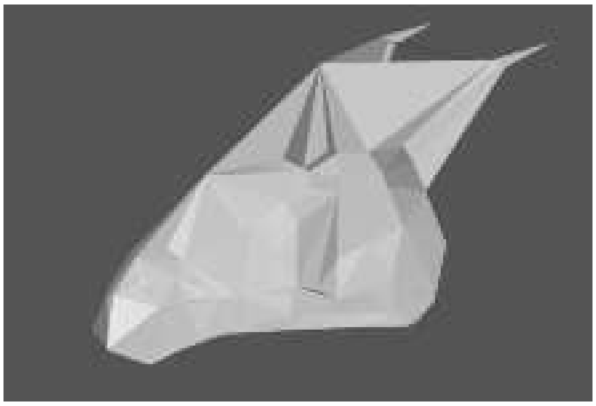
Alpha shapes give a “boundary” for unstructured points.



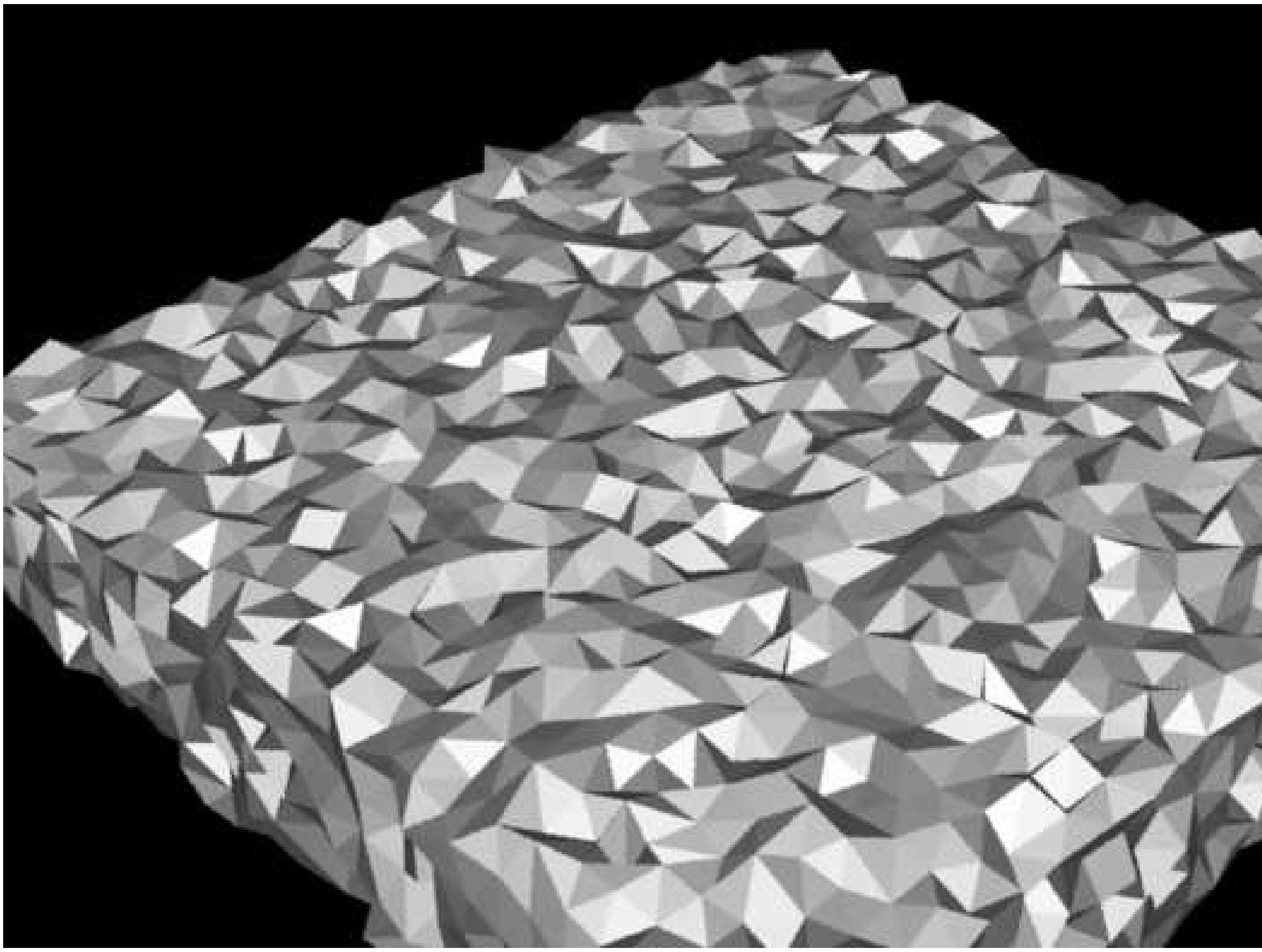
Smaller α



An "optimum" α



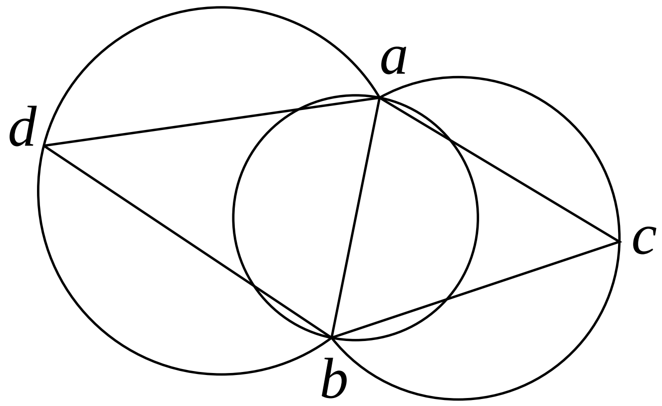
Larger α



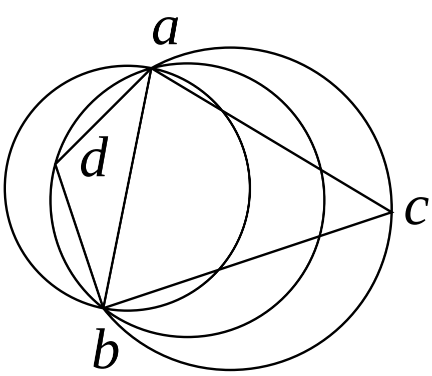
The alpha-shape boundary is chosen in a scale-independent way.

Alpha shapes: numerically

Checking if an edge $\{a, b\}$
can have an empty circle of radius r,
in intervals: radius(a, b) and
max$\{$radius(a, b, d), radius$(a, b, c)\}$



radius(a, b, c) and radius(a, b, d)



To check these cases, need:

- Delaunay circles of $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$,
- incircle test for the circle of $\{a, b\}$.

Centers of Delaunay balls

Given $p_0, p_1 \ldots p_k$,
we want:

1. c equidistant from all p_i;

2. c closest satisfying this.

Translate by p_0, and so
$(p_i - c) \cdot (p_i - c) = (p_0 - c) \cdot (p_0 - c)$
implies
$p_i \cdot c - p_i^2/2 = 0.$

We want

1. $[c\ -1/2] \cdot [p_i\ p_i^2] = 0$, and

2. $c = \sum_i \alpha_i p_i$

Apply GS to $p_1 \ldots, p_k, [0\ 1]$, getting $[n\ \eta]$ so that

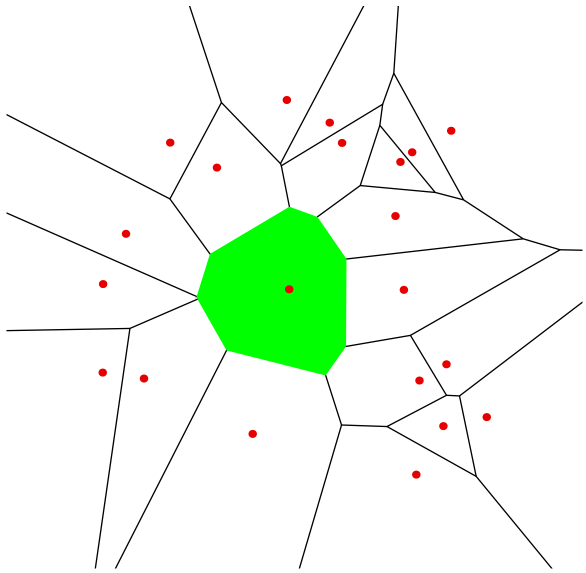
1. $[n\ \eta] \cdot [p_i\ p_i^2] = 0,\ i = 1 \ldots k$

2. $[0\ 1] = [n\ \eta] + \sum_i \beta_i [p_i\ p_i^2]$,

Volumes of Voronoi Regions

To compute the volume of a Voronoi cell C, recursively compute the volumes of its facets,

- $\mathsf{Vol}(C) = \frac{1}{d}\sum_F \mathsf{sdist}(c, F)\,\mathsf{Vol}(F)$,
- c the Delaunay center for sites giving C;
- $\mathsf{sdist}(c, F)$ = signed distance c to F.



Choice of c allows:

- sign of sdist() from incircle test;
- magnitude of sdist() from circumradii.

Scattered data interpolation

Given

sites s and values $f(s)$ for sites s,

get a "reasonable" estimate for $f(x)$,
x not a site.

- assume smoothness of f,
  suggesting linear interpolation;

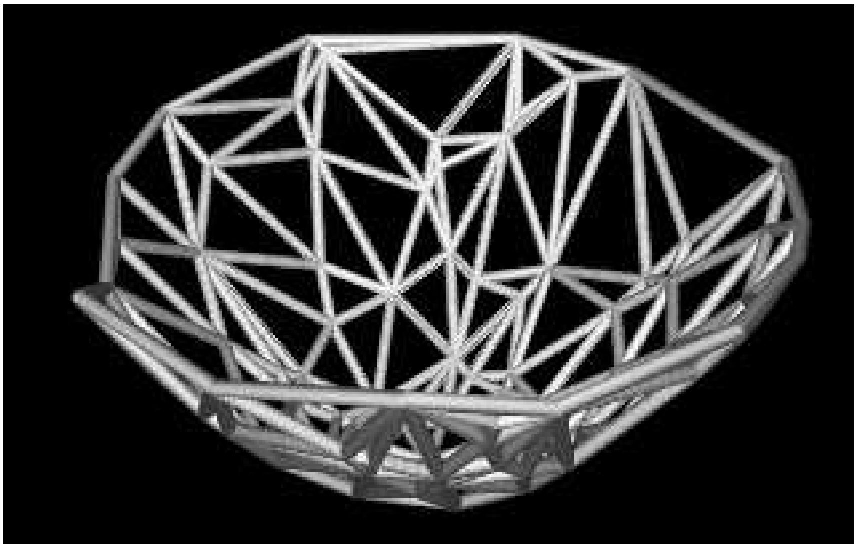
- magnitude of $f(x)$ bounded by magnitude of $f(s)$,
  suggesting convexity.

So, find x as a convex combination of s,
$f(x)$ as the same combination of $f(s)$.

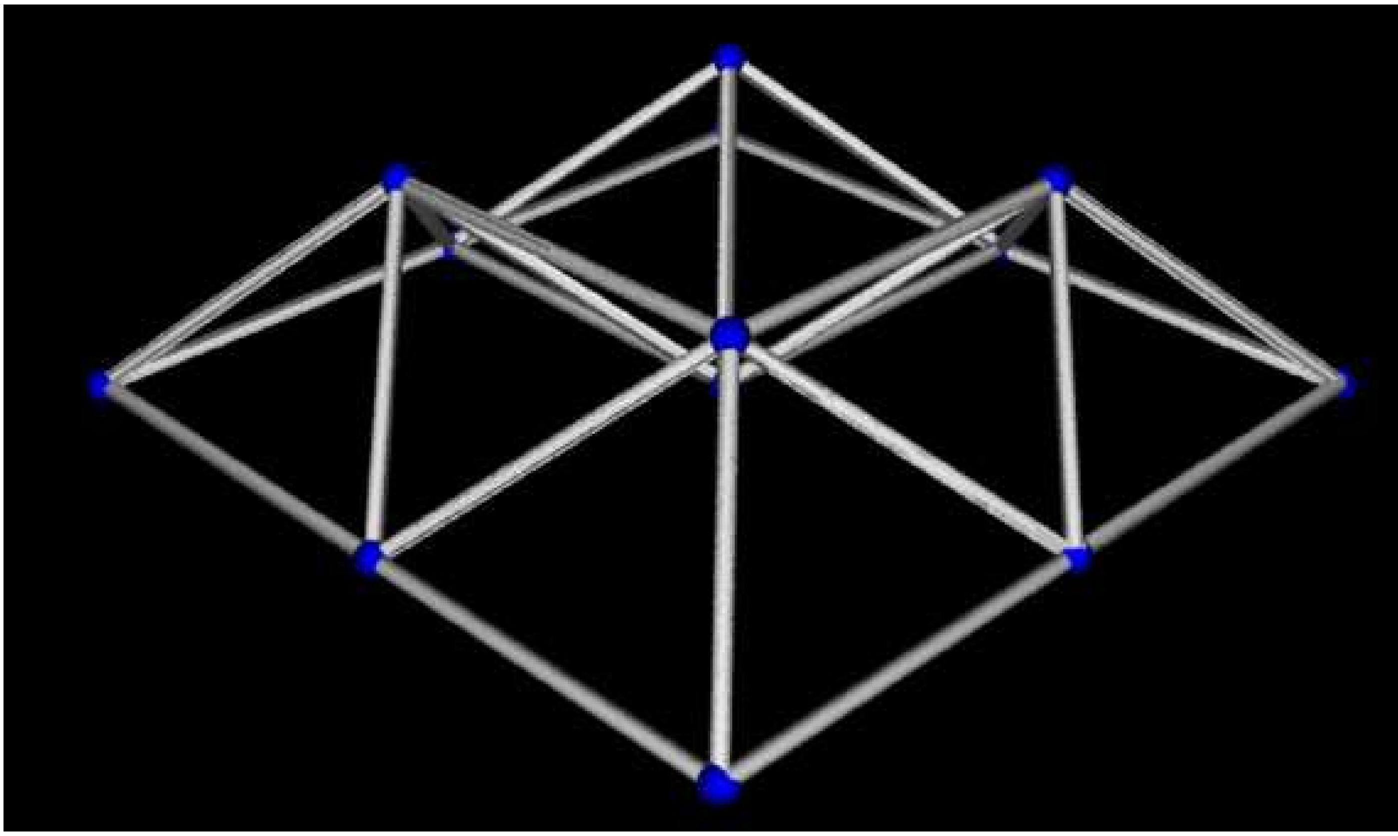
What convex combination?

Gale transforms describe whole space of possibilities.

When f is convex,
lower hull of $[s\ f(s)]$ mins maximum error.



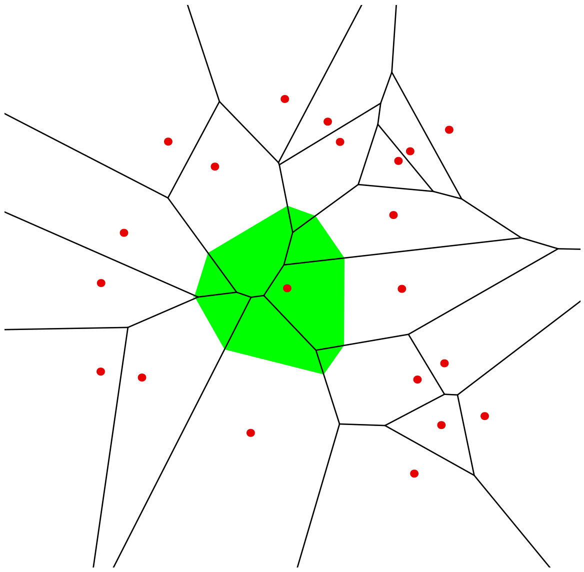
Isotropically, assume Hessian is the identity; get piece-wise linear in Delaunay triangles.



Natural Neighbor Interpolation

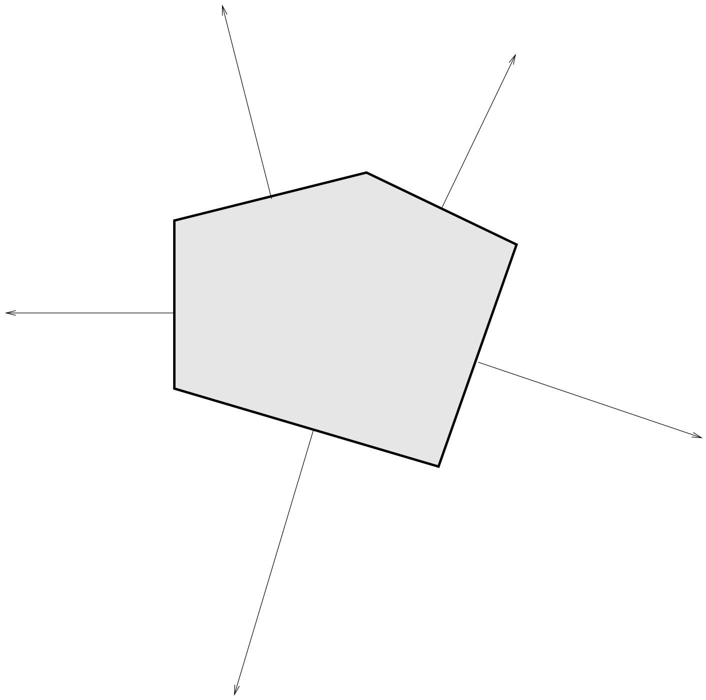
Compute V. diagram of sites;
add x;
Let w_s be the proportion of V. region of x that belonged to s.

Then $x = \sum_s w_s s$. (!)



natural consequence of:
P a polytope, F a facet, n_F normal to F, $v_F = \mathrm{Vol}(F)$,

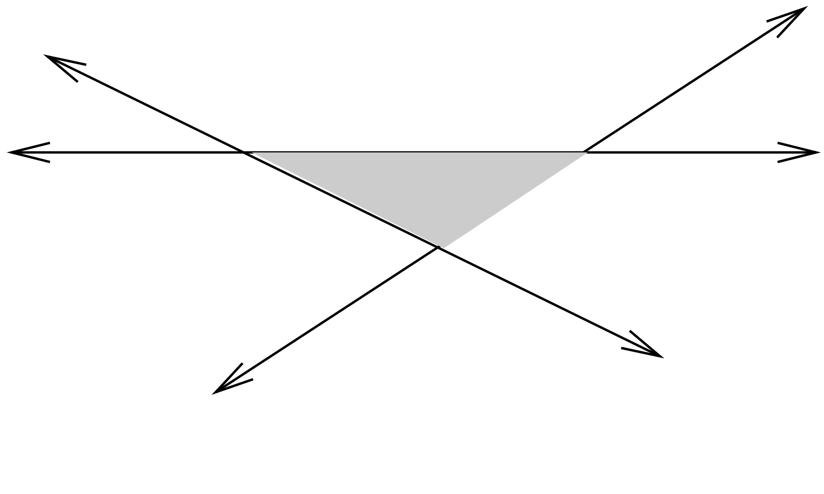
$$0 = \sum_F n_F v_F.$$



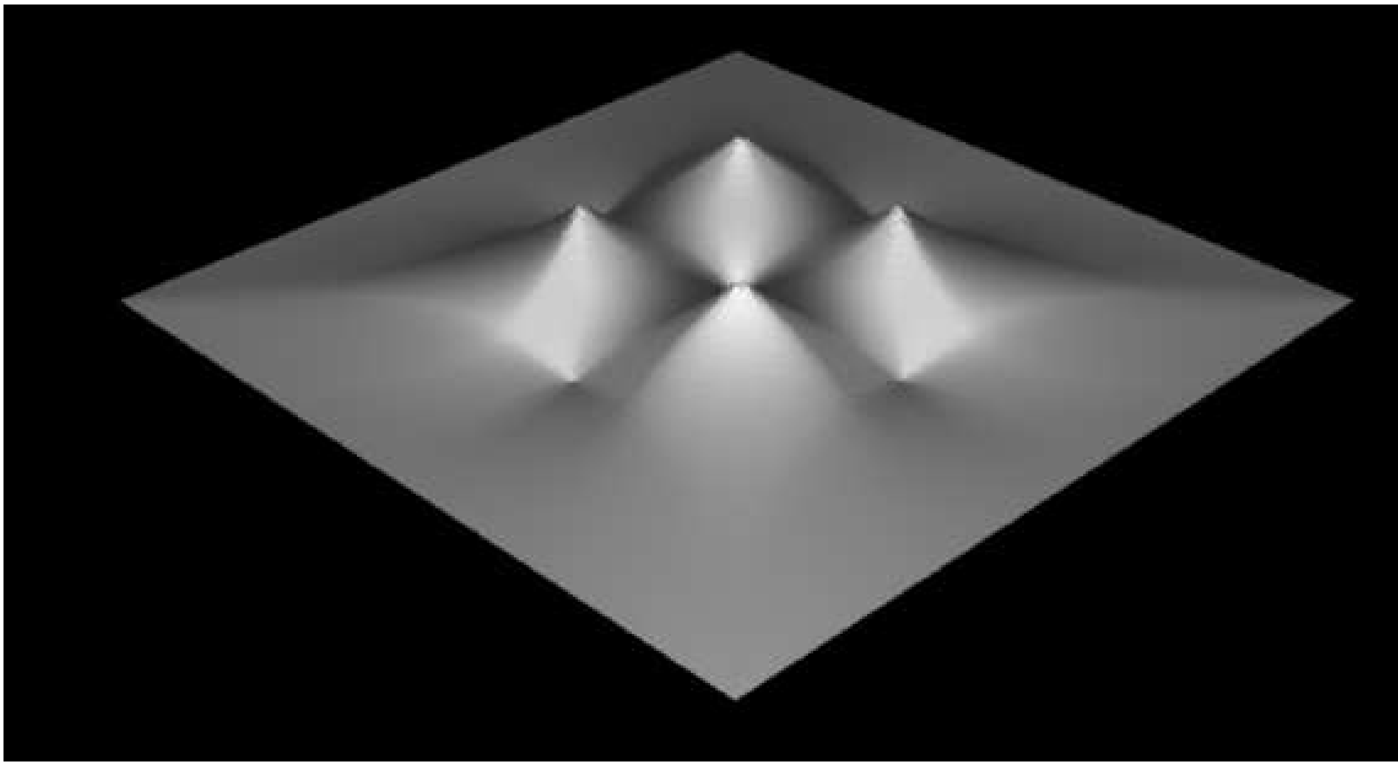
Now consider halfspaces
comprising points $[v \ z]$ with $z \geq v \cdot s - s^2/2$;

Their intersection $\equiv$ to V. diagram of sites;
Add halfspace $z \leq v \cdot x - x^2/2$;

Normals are $[s \ \ -1]$ and $[x \ \ -1]$;
weighted sum of normals =0 implies x a convex combination.



Basic NN interpolant is C^1 except at the sites: weights obtained by integrating a continuous function.



To get something smooth at the sites, must give up (a little) on convexity.

Sibson proposed: make interpolation of $x \cdot x$ perfect.

My (new?) way to do this:

Suppose $x = \sum_s \alpha_s s$,
and for s with $\alpha_s \neq 0$,
$s = \beta x + \sum_t \alpha'_t t$, so $x = \frac{1}{\beta}\left(s - \sum_t \alpha'_t t\right)$.

By convexity, we have

- $x^2 \leq \sum_s \alpha_s s^2$,
- $x^2 \geq \frac{1}{\beta}\left(s^2 - \sum_t \alpha'_t t^2\right)$

Use a linear combination of the RHS to get x and x^2 as linear combinations of neighbors of x:

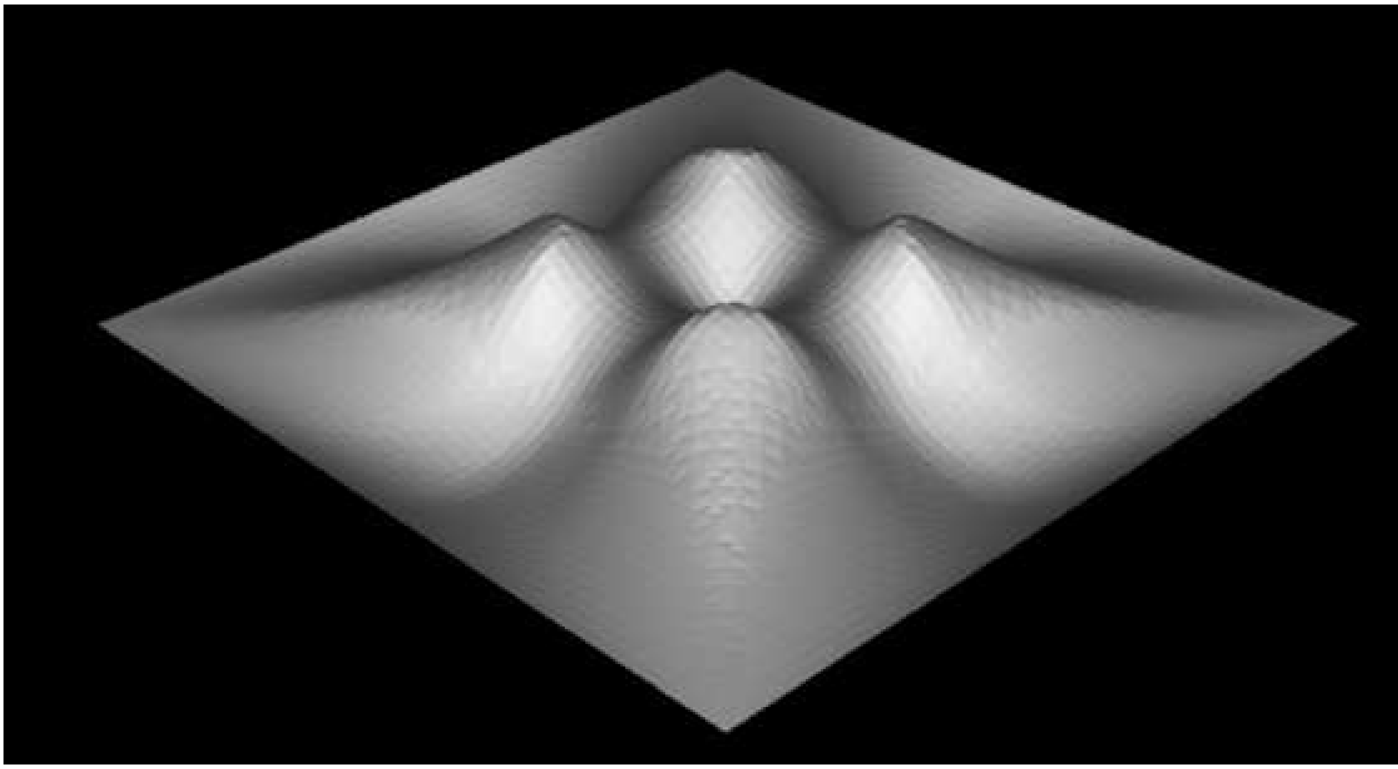
$x = \sum_t \gamma_t^s t$, and also $x^2 = \sum_t \gamma_t^s t^2$. Now combine expressions over neighbors s:

$$x = \sum_{s,t} \gamma_t^s \alpha_s t,$$

satisfied analogously by x^2.

A smoother NN Interpolant

The resulting interpolant is smoother; what are its properties?



A (different) convex combination

Apply sum-of-normals result to V region of x; Normals are $s - x$, so $\sum_F v_F n_F = 0$ gives $\sum_F v_F(s - x) = 0$, or

$$\sum_F v_F s = \sum_F v_F x.$$

This is used in the "finite volume element method."

Concluding Remarks

- Sometimes generality yields simplicity;
- A fast smooth NN intepolator requires dynamic hull;
- Could use any regular triangulation for convex combinations.

here come my slides!!!!